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Hermite-Hadamard Fejér Inequalities for Fractional Integrals for Functions Whose Second-Order Mixed Derivatives are Coordinated Preinvex

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Abstract

The main aim of this article is to establish some new refinements of Hermite Hadmard type inequalities via coordinate preinvex functions for fractional integrals. Here we give special cases to our results.

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1 Introduction

In recent years, a great deal of attention has been laid by many researchers to the theory of convexity because of its great importance in various fields of pure and applied sciences. The theories of convex function and inequalities are closely related to each other. A very interesting and well known inequality that establishes this connection is Hermite-Hadamard inequality, given as:

Let $g : H \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $\gamma_1, \gamma_2 \in H$ with $\gamma_1 < \gamma_2$, then

$$g\left(\frac{\gamma_1+\gamma_2}{2}\right) \leq \frac{1}{\gamma_2-\gamma_1}\int\limits_{\gamma_1}^{\gamma_2} g(v)dv \leq \frac{g(\gamma_1)+g(\gamma_2)}{2}. \tag{1}$$

Weighted generalization of the inequality (1) was presented by Fejér [5] as follows:

$$g\left(\frac{\gamma_1+\gamma_2}{2}\right)\int\limits_{\gamma_1}^{\gamma_2} w(v)dv \leq \int\limits_{\gamma_1}^{\gamma_2} w(v)g(v)dv \leq \frac{g(\gamma_1)+g(\gamma_2)}{2}\int\limits_{\gamma_1}^{\gamma_2} w(v)dv, \tag{2}$$

where $w : [\gamma_1; \gamma_2] \to \mathbb{R}$ is non negative, integrable and symmetric about $v = \frac{\gamma_1+\gamma_2}{2}$.

The concept of invex sets was given by T. Antczak [2].

Definition 1.1. *A set $H \subseteq \mathbb{R}^n$ is invex with respect to the map $\zeta : H \times H \to \mathbb{R}^n$ if for every $\gamma_1, \gamma_2 \in H$ and $t \in [0, 1]$, $\gamma_2 + t\zeta(\gamma_1, \gamma_2) \in H$. The invex set H is also called an ζ-connected set.*

Corollary 1.1. *Every convex set is an invex set but its converse is not true.*

In 1998, Weir and Mond [13], defined preinvex functions, which are generalization of convex functions.

Definition 1.2. *Let $H \subseteq \mathbb{R}^n$ be an invex set and a function $g : H \to \mathbb{R}$ is said to be preinvex w.r.t ζ if $\forall$ $\gamma_1, \gamma_2 \in H$,*

$$g(\gamma_1 + t\zeta(\gamma_1, \gamma_2)) \leq tg(\gamma_2) + (1-t)g(\gamma_1).$$

If $\zeta(\gamma_1, \gamma_2) = \gamma_1 - \gamma_2$, then in the classical sense, the preinvex function becomes a convex functions. A function g is called preincave iff its negative is preinvex.

Definition 1.3. [10] *Let $H_1 \times H_2$ be invex set with respect to $\zeta : H_1 \times H_1 \to \mathbb{R}^n$ and $\xi : H_2 \times H_2 \to \mathbb{R}^n$. A function $g : H_1 \times H_2 \to \mathbb{R}$ is said to be preinvex in coordinates if for every $(x, y), (u, v) \in H_1 \times H_2$ and $t \in [0, 1]$, we have*

$$g(u + t\zeta(x, u), v + t\xi(y, v)) \leq (1-t)g(u, v) + tg(x, y). \tag{3}$$

Definition 1.4. [9] *Let $H_1 \times H_2$ be invex set with respect to $\zeta : H_1 \times H_1 \to \mathbb{R}^n$ and $\xi : H_2 \times H_2 \to \mathbb{R}^n$. A function $g : H_1 \times H_2 \to \mathbb{R}$ is said to be preinvex in coordinates if for every $(x, y), (x, u), (u, y), (u, v) \in H_1 \times H_2$ and $\lambda, t \in [0, 1]$, we have*

$$g(u+\lambda\zeta(x, u), v+t\xi(y, v)) \leq (1-\lambda)(1-t)g(u, v)+(1-\lambda)tg(u, y)+(1-t)\lambda g(x, v)+\lambda tg(x, y). \tag{4}$$

Corollary 1.2. *If we substitute $\lambda = t$ in (4), then (4) reduces to (3).*

For more details about invex set and preinvex function, (See [11]).

Dragomir in [4], established the following result:

Theorem 1.1. *suppose that $g : \Delta = [\gamma_1, \gamma_2] \times [\gamma_3, \gamma_4] \to \mathbb{R}$ is a coordinated convex function on Δ. Then one has the following inequalities:*

$$\begin{aligned} g\left(\frac{\gamma_1+\gamma_2}{2}, \frac{\gamma_3+\gamma_4}{2}\right) &\leqslant \frac{1}{2}\left[\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2} g\left(x, \frac{\gamma_3+\gamma_4}{2}\right)dx + \frac{1}{\gamma_4-\gamma_3}\int_{\gamma_3}^{\gamma_4} g\left(\frac{\gamma_1+\gamma_2}{2}, y\right)dy\right] \\ &\leqslant \frac{1}{(\gamma_2-\gamma_1)(\gamma_4-\gamma_3)}\int_{\gamma_1}^{\gamma_2}\int_{\gamma_3}^{\gamma_4} g(x,y)dydx \\ &\leqslant \frac{1}{4}\left[\frac{1}{\gamma_2-\gamma_1}\left(\int_{\gamma_1}^{\gamma_2} g(x,\gamma_3)\,\delta x + \int_{\gamma_1}^{\gamma_2} g(x,\gamma_4)\,dx\right)\right. \\ &\quad \left. + \frac{1}{\gamma_4-\gamma_3}\left(\int_{\gamma_3}^{\gamma_4} g(\gamma_1,y)\,\delta y + \int_{\gamma_3}^{\gamma_4} g(\gamma_2,y)\,dy\right)\right] \\ &\leqslant \frac{g(\gamma_1,\gamma_3)+g(\gamma_1,\gamma_4)+g(\gamma_2,\gamma_3)+g(\gamma_2,\gamma_4)}{4}. \end{aligned}$$

Alomari and Darus [1] extended the Hermite-Hadamard inequality to Fejér inequality as:.

Theorem 1.2. *suppose that $g : \Delta = [\gamma_1, \gamma_2] \times [\gamma_3, \gamma_4] \to \mathbb{R}$ be a coordinated convex function on Δ. Then one has the following inequalities:*

$$\begin{aligned} g\left(\frac{\gamma_1+\gamma_2}{2}, \frac{\gamma_3+\gamma_4}{2}\right)\int_{\gamma_1}^{\gamma_2}\int_{\gamma_3}^{\gamma_4} p(x,y)dydx &\leqslant \int_{\gamma_1}^{\gamma_2}\int_{\gamma_3}^{\gamma_4} g(x,y)p(x,y)dydx \\ &\leqslant \frac{g(\gamma_1,\gamma_3)+g(\gamma_1,\gamma_4)+g(\gamma_2,\gamma_3)+g(\gamma_2,\gamma_4)}{4} \\ &\quad \times \int_{\gamma_1}^{\gamma_2}\int_{\gamma_3}^{\gamma_4} p(x,y)dydx, \end{aligned}$$

where $p : [\gamma_1, \gamma_2] \times [\gamma_3, \gamma_4] \to \mathbb{R}$ is positive, integrable and symmetric about $x = \frac{\gamma_1+\gamma_2}{2}$ and $y = \frac{\gamma_3+\gamma_4}{2}$.

Latif and Dragomir [8] proposed the following results.

Lemma 1.1. *Let $g : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [\gamma_1, \gamma_2] \times [\gamma_3, \gamma_4]$ with $\gamma_1 < \gamma_2, \gamma_3 < \gamma_4$. If $g_{st} \in L(\Delta)$, then the following identity holds:*

$$\begin{aligned} & g\left(\frac{\gamma_1+\gamma_2}{2}, \frac{\gamma_3+\gamma_4}{2}\right) + \frac{1}{(\gamma_2-\gamma_1)(\gamma_4-\gamma_3)}\int_{\gamma_1}^{\gamma_2}\int_{\gamma_3}^{\gamma_4} g(x,y)\,dydx \\ & -\frac{1}{(\gamma_2-\gamma_1)}\int_{\gamma_1}^{\gamma_2} g\left(x, \frac{\gamma_3+\gamma_4}{2}\right)dx - \frac{1}{(\gamma_4-\gamma_3)}\int_{\gamma_3}^{\gamma_4} g\left(\frac{\gamma_1+\gamma_2}{2}, y\right)dy \\ = \;& (\gamma_2-\gamma_1)(\gamma_4-\gamma_3)\int_0^1\int_0^1 H(s,t)g_{st}(t\alpha+(1-t)\gamma_2, s\gamma+(1-s)\gamma_4)dsdt, \end{aligned} \tag{5}$$

where

$$H(s,t) = \begin{cases} ts, & (s,t) \in \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right), \\ t(s-1), & (s,t) \in \left[0, \frac{1}{2}\right] \times \left(\frac{1}{2}, 1\right], \\ s(t-1), & (s,t) \in \left(\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right], \\ (t-1)(s-1), & (s,t) \in \left(\frac{1}{2}, 1\right] \times \left(\frac{1}{2}, 1\right]. \end{cases}$$

Theorem 1.3. *Let $g : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [\gamma_1, \gamma_2] \times [\gamma_3, \gamma_4]$ with $\gamma_1 < \gamma_2, \gamma_3 < \gamma_4$. If $|g_{st}|$ is convex on the coordinates on Δ, then the following inequality holds:*

$$\begin{aligned}
&\left| g\left(\frac{\gamma_1+\gamma_2}{2}, \frac{\gamma_3+\gamma_4}{2}\right) + \frac{1}{(\gamma_2-\gamma_1)(\gamma_4-\gamma_3)} \int_{\gamma_1}^{\gamma_2}\int_{\gamma_3}^{\gamma_4} g(x,y)\,dydx \right.\\
&\left. - \frac{1}{(\gamma_2-\gamma_1)} \int_{\gamma_1}^{\gamma_2} g\left(x, \frac{\gamma_3+\gamma_4}{2}\right) dx - \frac{1}{(\gamma_4-\gamma_3)} \int_{\gamma_3}^{\gamma_4} g\left(\frac{\gamma_1+\gamma_2}{2}, y\right) dy \right|\\
\leq\ & \frac{(\gamma_2-\gamma_1)(\gamma_4-\gamma_3)}{64} \left[|g_{st}(\gamma_1,\gamma_3)| + |g_{st}(\gamma_1,\gamma_4)| + |g_{st}(\gamma_2,\gamma_3)| + |g_{st}(\gamma_2,\gamma_4)| \right]. \qquad (6)
\end{aligned}$$

Theorem 1.4. *Let $g : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [\gamma_1, \gamma_2] \times [\gamma_3, \gamma_4]$ with $\gamma_1 < \gamma_2, \gamma_3 < \gamma_4$. If $|g_{st}|^q$ is convex on the coordinates on Δ and $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then following inequality holds:*

$$\begin{aligned}
&\left| g\left(\frac{\gamma_1+\gamma_2}{2}, \frac{\gamma_3+\gamma_4}{2}\right) + \frac{1}{(\gamma_2-\gamma_1)(\gamma_4-\gamma_3)} \int_{\gamma_1}^{\gamma_2}\int_{\gamma_3}^{\gamma_4} g(x,y)\,dydx \right.\\
&\left. - \frac{1}{(\gamma_2-\gamma_1)} \int_{\gamma_1}^{\gamma_2} g\left(x, \frac{\gamma_3+\gamma_4}{2}\right) dx - \frac{1}{(\gamma_4-\gamma_3)} \int_{\gamma_3}^{\gamma_4} g\left(\frac{\gamma_1+\gamma_2}{2}, y\right) dy \right|\\
\leq\ & \frac{(\gamma_2-\gamma_1)(\gamma_4-\gamma_3)}{4(p+1)^{\frac{2}{p}}} \left(\frac{1}{4} \left[|g_{st}(\gamma_1,\gamma_3)|^q + |g_{st}(\gamma_1,\gamma_4)|^q + |g_{st}(\gamma_2,\gamma_3)|^q + |g_{st}(\gamma_2,\gamma_4)|^q \right] \right)^{\frac{1}{q}} \quad (7)
\end{aligned}$$

Theorem 1.5. *Let $g : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [\gamma_1, \gamma_2] \times [\gamma_3, \gamma_4]$ with $\gamma_1 < \gamma_2, \gamma_3 < \gamma_4$. If $|g_{st}|^q$ is convex on the coordinates on Δ and $q \geq 1$, then following inequality holds:*

$$\begin{aligned}
&\left| g\left(\frac{\gamma_1+\gamma_2}{2}, \frac{\gamma_3+\gamma_4}{2}\right) + \frac{1}{(\gamma_2-\gamma_1)(\gamma_4-\gamma_3)} \int_{\gamma_1}^{\gamma_2}\int_{\gamma_3}^{\gamma_4} g(x,y)\,dydx \right.\\
&\left. - \frac{1}{(\gamma_2-\gamma_1)} \int_{\gamma_1}^{\gamma_2} g\left(x, \frac{\gamma_3+\gamma_4}{2}\right) dx - \frac{1}{(\gamma_4-\gamma_3)} \int_{\gamma_3}^{\gamma_4} g\left(\frac{\gamma_1+\gamma_2}{2}, y\right) dy \right|\\
\leq\ & \frac{(\gamma_2-\gamma_1)(\gamma_4-\gamma_3)}{16} \left(\frac{1}{4} \left[|g_{st}(\gamma_1,\gamma_3)|^q + |g_{st}(\gamma_1,\gamma_4)|^q + |g_{st}(\gamma_2,\gamma_3)|^q + |g_{st}(\gamma_2,\gamma_4)|^q \right] \right)^{\frac{1}{q}} \quad (8)
\end{aligned}$$

Latif and Dragomir in [9], developed the following results.

Lemma 1.2. *Let $H_1 \times H_2 \to \mathbb{R}$ be an open convex subset of $\mathbb{R}^2$ with respect to the mappings $\zeta : H_1 \times H_1 \to \mathbb{R}$ and $\xi : H_2 \times H_2 \to \mathbb{R}$. Suppose $g : H_1 \times H_2 \to \mathbb{R}$ be twice partial differentiable mapping such that*

$$\frac{\partial^2 g}{\partial\lambda\partial t} \in L_1([\gamma_1, \gamma_1 + \zeta(\gamma_2,\gamma_1)] \times [\gamma_3, \gamma_3 + \xi(\gamma_4,\gamma_3)]),$$

where $\zeta(\gamma_2,\gamma_1) \neq 0, \xi(\gamma_4,\gamma_3) \neq 0$, where $\gamma_1, \gamma_2 \in H_1$ and $\gamma_3, \gamma_4 \in H_2$. Then the following equality holds:

$$\begin{aligned}
&\frac{1}{4}[g(\gamma_1,\gamma_3) + g(\gamma_1, \gamma + \xi(\gamma_4,\gamma_3)) + g(\gamma_1 + \zeta(\gamma_2,\gamma_1), \gamma_3) + g(\gamma_1 + \zeta(\gamma_2,\gamma_1), \gamma_3 + \xi(\gamma_4,\gamma_3))])\\
&+ \frac{1}{\zeta(\gamma_2,\gamma_1)\xi(\gamma_4,\gamma_3)} \int_{\gamma_1}^{\gamma_1+\zeta(\gamma_2,\gamma_1)} \int_{\gamma_3}^{\gamma_3+\xi(\gamma_4,\gamma_3)} g(x,y)dxdy - D\\
=\ & \frac{\zeta(\gamma_2,\gamma_1)\xi(\gamma_4,\gamma_3)}{4} \int_0^1\int_0^1 (1-2\lambda)(1-2t) \frac{\partial^2 g}{\partial t\partial\lambda}(\gamma_1 + \lambda\zeta(\gamma_2,\gamma_1), \gamma_3 + t\xi(\gamma_4,\gamma_3))d\lambda dt, \qquad (9)
\end{aligned}$$

where

$$D = \frac{1}{2\zeta(\gamma_2,\gamma_1)}\int_{\gamma_1}^{\gamma_1+\zeta(\gamma_2,\gamma_1)}[g(x,\gamma_3)+g(x,\gamma_3+\xi(\gamma_4,\gamma_3))]dx + \frac{1}{2\xi(\gamma_4,\gamma_3)}\int_{\gamma_3}^{\gamma_3+\xi(\gamma_4,\gamma_3)} g(\gamma_1,y)+g(\gamma_1+\zeta(\gamma_2,\gamma_1),y)]dy.$$

In [6], C. Haisong presented the new Hermite-Hadamard type inequality for coordinate convex function. In [3], A. Babakhani established the Hermite-Hadamard-Fejér inequalities for coordinated convex functions involving fractional integral .

Recently, fractional calculus has proved to be a powerful tool in different fields of sciences. Because of the wide application of fractional calculus and Hermite-Hadamard inequalities, researchers have extended their work on Hermite-Hadamard inequalities in fractional domain.

In [12], Sarikaya et al. presented Hermite-Hadamard's inequalities for fractional integral as follows.

Theorem 1.6. *Let $g:[\gamma_1,\gamma_2]\subseteq\mathbb{R}\to\mathbb{R}$ with $0\le\gamma_1<\gamma_2$ and $g\in L[\gamma_1,\gamma_2]$. If g is positive and convex function on $[\gamma_1,\gamma_2]$, then the following inequalities for fractional integrals hold*

$$g\left(\frac{\gamma_1+\gamma_2}{2}\right)\le\frac{\Gamma(\rho+1)}{2(\gamma_2-\gamma_1)^\rho}\left[J^\rho_{\gamma_2}g(\gamma_1)+J^\rho_{\gamma_1}g(\gamma_2)\right]\le\frac{g(\gamma_1)+g(\gamma_2)}{2}.$$

with $\rho>0$. Here, the symbols $J^\rho_{\gamma_1}$ and $J^\rho_{\gamma_2}$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\rho\in\mathbb{R}^+$ that are defined in [7]

$$J^\rho_{\gamma_1}g(x)=\frac{1}{\Gamma(\rho)}\int_{\gamma_1}^{x}(x-t)^{\rho-1}g(t)dt,\ 0\le\gamma_1<x\le\gamma_2,$$

and

$$J^\rho_{\gamma_2}g(x)=\frac{1}{\Gamma(\rho)}\int_{x}^{\gamma_2}(t-x)^{\rho-1}g(t)dt,\ 0\le\gamma_1\le x<\gamma_2.$$

In the case of $\rho=1$, the fractional integral reduces to the classical integral.

In this section, we present two new Hermite-Hadamard-Fejér identities for functions second-order mixed derivatives are coordinated preinvex for fractional integrals. Using the new identities, we obtain some new weighted estimates connected with the left and right hand side of the Hermite-Hadamard-Fejér type inequalities for coordinated preinvex functions.

2 Main Results

Throughout this section, we will let $\sup_{(s,t)\in(0,1)\times(0,1)}|w(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))|=\|w\|_\infty$.

Lemma 2.1. *Let $K_1 \times K_2 \subseteq \mathbb{R}^2$ be an open invex subset with respect to $\eta : K_1 \times K_1 \to \mathbb{R}^2$ and $\xi : K_2 \times K_2 \to \mathbb{R}^2$. Suppose $f : K_1 \times K_2 \to \mathbb{R}^2$ is a twice differentiable mapping on $K_1 \times K_2$ such that $f_{st} \in L\left([\gamma_1, \gamma_1 + \eta(\gamma_2, \gamma_1)] \times ([\gamma_3, \gamma_3 + \xi(\gamma_4, \gamma_3)])\right)$ where $\eta(\gamma_2, \gamma_1) > 0$ and $\xi(\gamma_4, \gamma_3) > 0$. If $w : ([\gamma_1, \gamma_1 + \eta(\gamma_2, \gamma_1)] \times ([\gamma_3, \gamma_3 + \xi(\gamma_4, \gamma_3)])) \to [0, \infty) \times [0, \infty)$ is an integral mapping, then for every $\gamma_1, \gamma_2 \in K_1$ and $\gamma_3, \gamma_4 \in K_2$, the following equality holds:*

$$
\begin{aligned}
I = {} & f\left(\gamma_1 + \frac{1}{2}\eta(\gamma_2, \gamma_1), \gamma_3 + \frac{1}{2}\xi(\gamma_4, \gamma_3)\right) \Big[I^{\rho,\sigma}_{\left(\gamma_1 + \frac{1}{2}\eta(\gamma_2,\gamma_1)\right)-,\left(\gamma_3 + \frac{1}{2}\xi(\gamma_4,\gamma_3)\right)-} w(\gamma_1, \gamma_3) \\
& + I^{\rho,\sigma}_{\left(\gamma_1 + \frac{1}{2}\eta(\gamma_2,\gamma_1)\right)+,\left(\gamma_3 + \frac{1}{2}\xi(\gamma_4,\gamma_3)\right)-} w(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3) \\
& + I^{\rho,\sigma}_{\left(\gamma_1 + \frac{1}{2}\eta(\gamma_2,\gamma_1)\right)-,\left(\gamma_3 + \frac{1}{2}\xi(\gamma_4,\gamma_3)\right)+} w(\gamma_1, \gamma_3 + \xi(\gamma_4, \gamma_3)) \\
& + I^{\rho,\sigma}_{\left(\gamma_1 + \frac{1}{2}\eta(\gamma_2,\gamma_1)\right)+,\left(\gamma_3 + \frac{1}{2}\xi(\gamma_4,\gamma_3)\right)+} w(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3 + \xi(\gamma_4, \gamma_3)) \Big] \\
& - \Big[I^{\rho,\sigma}_{\left(\gamma_1 + \frac{1}{2}\eta(\gamma_2,\gamma_1)\right)-,\left(\gamma_3 + \frac{1}{2}\xi(\gamma_4,\gamma_3)\right)-} f(\gamma_1, \gamma_3 + \frac{1}{2}\xi(\gamma_4, \gamma_3))\, w(\gamma_1, \gamma_3) \\
& + I^{\rho,\sigma}_{\left(\gamma_1 + \frac{1}{2}\eta(\gamma_2,\gamma_1)\right)+,\left(\gamma_3 + \frac{1}{2}\xi(\gamma_4,\gamma_3)\right)-} f(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3 + \frac{1}{2}\xi(\gamma_4, \gamma_3))\, w(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3) \\
& + I^{\rho,\sigma}_{\left(\gamma_1 + \frac{1}{2}\eta(\gamma_2,\gamma_1)\right)-,\left(\gamma_3 + \frac{1}{2}\xi(\gamma_4,\gamma_3)\right)+} f(\gamma_1, \gamma_3 + \frac{1}{2}\xi(\gamma_4, \gamma_3))\, w(\gamma_1, \gamma_3 + \xi(\gamma_4, \gamma_3)) \\
& + I^{\rho,\sigma}_{\left(\gamma_1 + \frac{1}{2}\eta(\gamma_2,\gamma_1)\right)+,\left(\gamma_3 + \frac{1}{2}\xi(\gamma_4,\gamma_3)\right)+} f(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3 + \frac{1}{2}\xi(\gamma_4, \gamma_3)\, w(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3 + \xi(\gamma_4, \gamma_3)) \Big] \\
& - \Big[I^{\rho,\sigma}_{\left(\gamma_1 + \frac{1}{2}\eta(\gamma_2,\gamma_1)\right)-,\left(\gamma_3 + \frac{1}{2}\xi(\gamma_4,\gamma_3)\right)-} f(\gamma_1 + \frac{1}{2}\eta(\gamma_2, \gamma_1), \gamma_3)\, w(\gamma_1, \gamma_3) \\
& + I^{\rho,\sigma}_{\left(\gamma_1 + \frac{1}{2}\eta(\gamma_2,\gamma_1)\right)+,\left(\gamma_3 + \frac{1}{2}\xi(\gamma_4,\gamma_3)\right)-} f(\gamma_1 + \frac{1}{2}\eta(\gamma_2, \gamma_1), \gamma_3)\, w(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3) \\
& + I^{\rho,\sigma}_{\left(\gamma_1 + \frac{1}{2}\eta(\gamma_2,\gamma_1)\right)-,\left(\gamma_3 + \frac{1}{2}\xi(\gamma_4,\gamma_3)\right)+} f(\gamma_1 + \frac{1}{2}\eta(\gamma_2, \gamma_1), \gamma_3 + \xi(\gamma_4, \gamma_3))\, w(\gamma_1, \gamma_3 + \xi(\gamma_4, \gamma_3)) \\
& + I^{\rho,\sigma}_{\left(\gamma_1 + \frac{1}{2}\eta(\gamma_2,\gamma_1)\right)+,\left(\gamma_3 + \frac{1}{2}\xi(\gamma_4,\gamma_3)\right)+} f(\gamma_1 + \frac{1}{2}\eta(\gamma_2, \gamma_1), \gamma_3 + \xi(\gamma_4, \gamma_3))\, w(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3 + \xi(\gamma_4, \gamma_3)) \Big] \\
& + \Big[I^{\rho,\sigma}_{\left(\gamma_1 + \frac{1}{2}\eta(\gamma_2,\gamma_1)\right)-,\left(\gamma_3 + \frac{1}{2}\xi(\gamma_4,\gamma_3)\right)-} (fw)(\gamma_1, \gamma_3) \\
& + I^{\rho,\sigma}_{\left(\gamma_1 + \frac{1}{2}\eta(\gamma_2,\gamma_1)\right)+,\left(\gamma_3 + \frac{1}{2}\xi(\gamma_4,\gamma_3)\right)-} (fw)(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3) \\
& + I^{\rho,\sigma}_{\left(\gamma_1 + \frac{1}{2}\eta(\gamma_2,\gamma_1)\right)-,\left(\gamma_3 + \frac{1}{2}\xi(\gamma_4,\gamma_3)\right)+} (fw)(\gamma_1, \gamma_3 + \xi(\gamma_4, \gamma_3)) \\
& + I^{\rho,\sigma}_{\left(\gamma_1 + \frac{1}{2}\eta(\gamma_2,\gamma_1)\right)+,\left(\gamma_3 + \frac{1}{2}\xi(\gamma_4,\gamma_3)\right)+} (fw)(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3 + \xi(\gamma_4, \gamma_3)) \Big] \\
= {} & \frac{[\eta(\gamma_2, \gamma_1)]^{\rho+2} [\xi(\gamma_4, \gamma_3)]^{\sigma+2}}{\Gamma(\rho)\Gamma(\sigma)} \left(\int_0^1 \int_0^1 K(s,t) f_{st}(\gamma_1 + s\eta(\gamma_2, \gamma_1), \gamma_3 + t\xi(\gamma_4, \gamma_3)) ds dt \right),
\end{aligned} \tag{10}
$$

where

$$
K(s,t) = \begin{cases} \int_0^t A_1(s,v)\, dv, & (s,t) \in \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right), \\ -\int_0^t A_2(s,v)\, dv, & (s,t) \in \left[0, \frac{1}{2}\right] \times \left(\frac{1}{2}, 1\right], \\ -\int_t^1 A_3(s,v)\, dv, & (s,t) \in \left(\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right], \\ \int_t^1 A_4(s,v)\, dv, & (s,t) \in \left(\frac{1}{2}, 1\right] \times \left(\frac{1}{2}, 1\right], \end{cases}
$$

and

$$
\begin{aligned}
A_1(s,v) &= \int_0^s u^{\rho-1}v^{\sigma-1}w(\gamma_1+u\eta(\gamma_2,\gamma_1),\gamma_3+v\xi(\gamma_4,\gamma_3))du,\\
A_2(s,v) &= \int_s^1 (1-u)^{\rho-1}v^{\sigma-1}w(\gamma_1+u\eta(\gamma_2,\gamma_1),\gamma_3+v\xi(\gamma_4,\gamma_3))du,\\
A_3(s,v) &= \int_0^s u^{\rho-1}(1-v)^{\sigma-1}w(\gamma_1+u\eta(\gamma_2,\gamma_1),\gamma_3+v\xi(\gamma_4,\gamma_3))du,\\
A_4(s,v) &= \int_s^1 (1-u)^{\rho-1}(1-v)^{\sigma-1}w(\gamma_1+u\eta(\gamma_2,\gamma_1),\gamma_3+v\xi(\gamma_4,\gamma_3))du.
\end{aligned}
$$

Proof. Expressing the integral on R.H.S in term of four integrals

$$
\begin{aligned}
&\int_0^1\int_0^1 K(s,t)f_{st}(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dsdt\\
={}& \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} K(s,t)g_{st}(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dtds\\
&+\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} K(s,t)g_{st}(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dtds\\
&+\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 K(s,t)g_{st}(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dtds\\
&+\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 K(s,t)g_{st}(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dtds\\
={}& I_1+I_2+I_3+I_4. \qquad (11)
\end{aligned}
$$

For computing I_1, we will first consider

$$
\begin{aligned}
&\int_0^{\frac{1}{2}}\int_0^t A_1(s,v)\,dv\, f_{st}(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dt\\
={}& \frac{1}{\xi(\gamma_4,\gamma_3)}\left| f_s(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))\int_0^t A_1(s,v)\,dv\right|_0^{\frac{1}{2}}\\
&-\frac{1}{\xi(\gamma_4,\gamma_3)}\int_0^{\frac{1}{2}} A_1(s,t)\, f_s(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dt\\
={}& \frac{1}{\xi(\gamma_4,\gamma_3)} f_s\left(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3)\right)\int_0^{\frac{1}{2}} A_1(s,v)\,dv\\
&-\frac{1}{\xi(\gamma_4,\gamma_3)}\int_0^{\frac{1}{2}} A_1(s,t)\, f_s(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dt. \qquad (12)
\end{aligned}
$$

Integrating (12) w.r.t s, we will get I_1

$$\begin{aligned} I_1 = & \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\int_0^t A_1(s,v)\,dv\; f_{st}(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dtds \\ = & \frac{1}{\xi(\gamma_4,\gamma_3)}\int_0^{\frac{1}{2}} f_s\left(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3)\right)\int_0^{\frac{1}{2}} A_1(s,v)\,dvds \\ & -\frac{1}{\xi(\gamma_4,\gamma_3)}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} A_1(s,t)\,f_s(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dtds. \end{aligned} \tag{13}$$

After computing the outer integral in (13), we obtain

$$\begin{aligned} I_1 = & \frac{1}{\eta(\gamma_2,\gamma_1)\xi(\gamma_4,\gamma_3)}\left[f\left(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1),\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3)\right)\right. \\ & \times\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} s^{\rho-1}t^{\sigma-1}w(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))\,dsdt \\ & -\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} s^{\rho-1}t^{\sigma-1}f\left(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3)\right)w(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))\,dsdt \\ & -\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} s^{\rho-1}t^{\sigma-1}f\left(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3)\right)w(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dsdt \\ & \left.+\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} s^{\rho-1}t^{\sigma-1}f(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))\,w(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dsdt\right]. \end{aligned} \tag{14}$$

On substituting $x=\gamma_1+s\eta(\gamma_2,\gamma_1)$ and $y=\gamma_3+t\xi(\gamma_4,\gamma_3)$ in (14), we obtain

$$\begin{aligned} I_1 = & \frac{1}{[\eta(\gamma_2,\gamma_1)]^{\rho+1}[\xi(\gamma_4,\gamma_3)]^{\sigma+1}}\left[f\left(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1),\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3)\right)\right. \\ = & \left[\int_{\gamma_1}^{\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1)}\int_{\gamma_3}^{\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3)}(x-\gamma_1)^{\rho-1}(y-\gamma_3)^{\sigma-1}w(x,y)dxdy\right. \\ & -\int_{\gamma_1}^{\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1)}\int_{\gamma_3}^{\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3)}(x-\gamma_1)^{\rho-1}(y-\gamma_3)^{\sigma-1}f(x,\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3)\;w(x,y)dxdy \\ & -\int_{\gamma_1}^{\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1)}\int_{\gamma_3}^{\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3)}(x-\gamma_1)^{\rho-1}(y-\gamma_3)^{\sigma-1}f(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1),y)\;w(x,y)dxdy \\ & \left.+\int_{\gamma_1}^{\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1)}\int_{\gamma_3}^{\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3)}(x-\gamma_1)^{\rho-1}(y-\gamma_3)^{\sigma-1}f(x,y)\;w(x,y)dxdy\right]. \end{aligned} \tag{15}$$

Utilizing the definition of Riemann-Liouville fractional integrals in (15), we obtain

$$\begin{aligned} & I_1 \\ = & \frac{\Gamma(\rho)\Gamma(\sigma)}{[\eta(\gamma_2,\gamma_1)]^{\rho+1}[\xi(\gamma_4,\gamma_3)]^{\sigma+1}}\left[f(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1),\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3))\right. \\ & \times I^{\rho,\sigma}_{\left(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1)\right)-,\left(\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3)\right)-}w(\gamma_1,\gamma_3) \\ & -I^{\rho,\sigma}_{\left(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1)\right)-,\left(\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3)\right)-}f(\gamma_1,\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3))\;w(\gamma_1,\gamma_3) \\ & -I^{\rho,\sigma}_{\left(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1)\right)-,\left(\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3)\right)-}f(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1),\gamma_3)\;w(\gamma_1,\gamma_3) \\ & \left.+I^{\rho,\sigma}_{\left(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1)\right)-,\left(\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3)\right)-}(fw)(\gamma_1,\gamma_3)\right]. \end{aligned} \tag{16}$$

Similarly, we obtain

$$\begin{aligned} & I_2 \\ = & \frac{\Gamma(\rho)\Gamma(\sigma)}{[\eta(\gamma_2,\gamma_1)]^{\rho+1}[\xi(\gamma_4,\gamma_3)]^{\sigma+1}}\Big[-f(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1),\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3)) \\ & \times I^{\rho,\sigma}_{(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1))+,(\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3))-} w(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3) \\ & +I^{\rho,\sigma}_{(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1))+,(\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3))-} f(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3))\, w(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3) \\ & +I^{\rho,\sigma}_{(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1))+,(\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3))-} f(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1),\gamma_3)\, w(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3) \\ & -I^{\rho,\sigma}_{(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1))+,(\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3))-} (fw)(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3)\Big]. \end{aligned} \tag{17}$$

$$\begin{aligned} & I_3 \\ = & \frac{\Gamma(\rho)\Gamma(\sigma)}{[\eta(\gamma_2,\gamma_1)]^{\rho+1}[\xi(\gamma_4,\gamma_3)]^{\sigma+1}}\Big[-f(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1),\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3)) \\ & \times I^{\rho,\sigma}_{(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1))-,(\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3))+} w(\gamma_1,\gamma_3+\xi(\gamma_4,\gamma_3)) \\ & +I^{\rho,\sigma}_{(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1))-,(\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3))+} f(\gamma_1,\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3))\, w(\gamma_1,\gamma_3+\xi(\gamma_4,\gamma_3)) \\ & +I^{\rho,\sigma}_{(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1))-,(\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3))+} f(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1),\gamma_3+\xi(\gamma_4,\gamma_3))\, w(\gamma_1,\gamma_3+\xi(\gamma_4,\gamma_3)) \\ & -\ I^{\rho,\sigma}_{(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1))-,(\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3))+} (fw)(\gamma_1,\gamma_3+\xi(\gamma_4,\gamma_3))\Big]. \end{aligned} \tag{18}$$

$$\begin{aligned} & I_4 \\ = & \frac{\Gamma(\rho)\Gamma(\sigma)}{[\eta(\gamma_2,\gamma_1)]^{\rho+1}[\xi(\gamma_4,\gamma_3)]^{\sigma+1}}\Big[-f(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1),\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3)) \\ & \times\Big[I^{\rho,\sigma}_{(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1))+,(\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3))+} w(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3+\xi(\gamma_4,\gamma_3)) \\ & -I^{\rho,\sigma}_{(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1))+,(\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3))+} f(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3))\, w(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3+\xi(\gamma_4,\gamma_3)) \\ & -I^{\rho,\sigma}_{(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1))+,(\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3))+} f(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1),\gamma_3+\xi(\gamma_4,\gamma_3))\, w(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3+\xi(\gamma_4,\gamma_3)) \\ & +\ I^{\rho,\sigma}_{(\gamma_1+\frac{1}{2}\eta(\gamma_2,\gamma_1))+,(\gamma_3+\frac{1}{2}\xi(\gamma_4,\gamma_3))+} (fw)(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3+\xi(\gamma_4,\gamma_3))\Big]. \end{aligned} \tag{19}$$

On utilizing (16), (17), (18) and (19) in (11), we obtain (10). □

Corollary 2.1. *If $f = 1, \rho = \sigma = 1, \eta(\gamma_2,\gamma_1) = \gamma_2 - \gamma_1$ and $\xi(\gamma_4,\gamma_3) = (\gamma_4 - \gamma_3)$, we get (5), which is Lemma 1 in [8].*

Theorem 2.1. *Let $K_1 \times K_2 \subseteq \mathbb{R}^2$ be an open invex subset with respect to $\eta : K_1 \times K_1 \to \mathbb{R}^2$ and $\xi : K_2 \times K_2 \to \mathbb{R}^2$. Suppose $f : K_1 \times K_2 \to \mathbb{R}^2$ is a twice differentiable mapping on $K_1 \times K_2$ such that $f_{st} \in L([\gamma_1, \gamma_1 + \eta(\gamma_2,\gamma_1)] \times ([\gamma_3, \gamma_3 + \xi(\gamma_4,\gamma_3)]))$ where $\eta(\gamma_2,\gamma_1) > 0$ and $\xi(\gamma_4,\gamma_3) > 0$. If $w : ([\gamma_1, \gamma_1 + \eta(\gamma_2,\gamma_1)] \times ([\gamma_3, \gamma_3 + \xi(\gamma_4,\gamma_3)])) \to [0,\infty) \times [0,\infty)$ is an integral mapping and $|f_{st}|$ is preinvex function then for every $\gamma_1, \gamma_2 \in K_1$ and $\gamma_3, \gamma_4 \in K_2$, the following inequality holds:*

$$|I| \leq \frac{\|w\|_\infty}{\Gamma(\rho)\Gamma(\sigma)} \frac{[\eta(\gamma_2,\gamma_1)]^{\rho+2}[\xi(\gamma_4,\gamma_3)]^{\sigma+2}}{\rho\sigma(\rho+1)(\sigma+1)2^{\rho+\sigma+2}} [|f_{st}(\gamma_1,\gamma_3)| + |f_{st}(\gamma_1,\gamma_4)| + |f_{st}(\gamma_2,\gamma_3)| + |f_{st}(\gamma_2,\gamma_4)|]. \tag{20}$$

Proof. Taking modulus of both sides of (10), we get

$$|I| = \left| \frac{[\eta(\gamma_2,\gamma_1)]^{\rho+2}[\xi(\gamma_4,\gamma_3)]^{\sigma+2}}{\Gamma(\rho)\Gamma(\sigma)} \int_0^1 \left(\int_0^1 K(s,t) f_{st}(\gamma_1 + s\eta(\gamma_2,\gamma_1), \gamma_3 + t\xi(\gamma_4,\gamma_3)) ds \right) dt \right|. \tag{21}$$

f_{st} is coordinated preinvex function, i.e

$$\begin{aligned} & f_{st}(\gamma_1 + s\eta(\gamma_2,\gamma_1), \gamma_3 + t\xi(\gamma_4,\gamma_3)) \\ \leq\ & (1-s)(1-t) f_{st}(\gamma_1,\gamma_3) + (1-s)t f_{st}(\gamma_1,\gamma_4) + (1-t)s f_{st}(\gamma_2,\gamma_3) + st f_{st}(\gamma_2,\gamma_4). \end{aligned}$$

From preinvexity of $|f_{st}|$, we have from (21)

$$\begin{aligned} |I| \leq\ & \frac{[\eta(\gamma_2,\gamma_1)]^{\rho+2}[\xi(\gamma_4,\gamma_3)]^{\sigma+2}}{\Gamma(\rho)\Gamma(\sigma)} \int_0^1 \left(\int_0^1 |K(s,t)| \, (1-s)(1-t) |f_{st}(\gamma_1,\gamma_3)| + (1-s)t |f_{st}(\gamma_1,\gamma_4)| \right. \\ & \left. +(1-t)s |f_{st}(\gamma_2,\gamma_3)| + st |f_{st}(\gamma_2,\gamma_4)| \right) dt. \end{aligned} \tag{22}$$

After a straightforward computation, we obtain

$$\begin{aligned} & \|w\|_\infty \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left(\int_0^t \int_0^s v^{\sigma-1} u^{\rho-1} du dv \, [(1-s)(1-t) |f_{st}(\gamma_1,\gamma_3)| + (1-s)tf \, |_{st}(\gamma_1,\gamma_4)| \right. \\ & \left. +(1-t)s |f_{st}(\gamma_2,\gamma_3)| + st |f_{st}(\gamma_2,\gamma_4)|] \right) dsdt \\ & = \frac{\|w\|_\infty}{2^{\rho+\sigma+4}\rho\sigma} \left[\frac{(\rho+3)(\sigma+3)}{(\rho^2+3\rho+2)(\sigma^2+3\sigma+2)} |f_{st}(\gamma_1,\gamma_3)| + \frac{(\rho+3)}{(\rho^2+3\rho+2)(\sigma+2)} |f_{st}(\gamma_1,\gamma_4)| \right. \\ & \left. + \frac{(\sigma+3)}{(\rho+2)(\sigma^2+3\sigma+2)} |f_{st}(\gamma_2,\gamma_3)| + \frac{1}{(\rho+2)(\sigma+2)} |f_{st}(\gamma_2,\gamma_4)| \right]. \end{aligned} \tag{23}$$

$$\begin{aligned} & \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \left(\int_0^t \int_s^1 v^{\sigma-1}(1-u)^{\rho-1} du dv \, [(1-s)(1-t) |f_{st}(\gamma_1,\gamma_3)| + (1-s)t |f_{st}(\gamma_1,\gamma_4)| \right. \\ & \left. +(1-t)s |f_{st}(\gamma_2,\gamma_3)| + st |f_{st}(\gamma_2,\gamma_4)|] \right) dsdt \\ & = \frac{\|w\|_\infty}{2^{\rho+\sigma+4}\rho\sigma} \left[\frac{(\sigma+3)}{(\rho+2)(\sigma^2+3\sigma+2)} |f_{st}(\gamma_1,\gamma_3)| + \frac{1}{(\rho+2)(\sigma+2)} |f_{st}(\gamma_1,\gamma_4)| \right. \\ & \left. + \frac{(\rho+3)(\sigma+3)}{(\rho^2+3\rho+2)(\sigma^2+3\sigma+2)} |f_{st}(\gamma_2,\gamma_3)| + \frac{(\rho+3)}{(\rho^2+3\rho+2)(\sigma+2)} |f_{st}(\gamma_2,\gamma_4)| \right]. \end{aligned} \tag{24}$$

$$\begin{aligned} & \|w\|_\infty \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \left(\int_t^1 \int_0^s u^{\rho-1}(1-v)^{\sigma-1} du dv \, [(1-s)(1-t) |f_{st}(\gamma_1,\gamma_3)| + (1-s)tf \, |_{st}(\gamma_1,\gamma_4)| \right. \\ & \left. +(1-t)s |f_{st}(\gamma_2,\gamma_3)| + st |f_{st}(\gamma_2,\gamma_4)|] \right) dsdt \\ & = \frac{\|w\|_\infty}{2^{\rho+\sigma+4}\rho\sigma} \left[\frac{(\rho+3)}{(\rho^2+3\rho+2)(2+\sigma)} |f_{st}(\gamma_1,\gamma_3)| + \frac{(\rho+3)(\sigma+3)}{(\rho^2+3\rho+2)(\sigma^2+3\sigma+2)} |f_{st}(\gamma_1,\gamma_4)| \right. \\ & \left. + \frac{1}{(\rho+2)(\sigma+2)} |f_{st}(\gamma_2,\gamma_3)| + \frac{(\rho+3)}{(\rho+2)(\sigma^2+3\sigma+2)} |f_{st}(\gamma_2,\gamma_4)| \right]. \end{aligned} \tag{25}$$

$$\|w\|_\infty \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} \left(\int_t^1\int_s^1 (1-v)^{\sigma-1}(1-u)^{\rho-1}dudv\left[(1-s)(1-t)\left|f_{st}(\gamma_1,\gamma_3)\right| + (1-s)tf\left|_{st}(\gamma_1,\gamma_4)\right|\right.\right.$$
$$\left.\left. +(1-t)s\left|f_{st}(\gamma_2,\gamma_3)\right| + st\left|f_{st}(\gamma_2,\gamma_4)\right|\right) dsdt$$
$$= \frac{\|w\|_\infty}{2^{\rho+\sigma+4}\rho\sigma}\left[\frac{1}{(\rho+2)(\sigma+2)}\left|f_{st}(\gamma_1,\gamma_3)\right| + \frac{(3+\sigma)}{(\rho+2)(\sigma^2+3\sigma+2)}\left|f_{st}(\gamma_1,\gamma_4)\right|\right.$$
$$\left. +\frac{(\rho+3)}{(\rho^2+3\rho+2)(\sigma+2)}\left|f_{st}(\gamma_2,\gamma_3)\right| + \frac{(\rho+3)(\sigma+3)}{(\rho^2+3\rho+2)(\sigma^2+3\sigma+2)}\left|f_{st}(\gamma_2,\gamma_4)\right|\right]. \tag{26}$$

Using the results of (23), (24), (25) and (26) in (22), we get (20). □

Corollary 2.2. *If $f=1, \rho=\sigma=1, \eta(\gamma_2,\gamma_1)=\gamma_2-\gamma_1$ and $\xi(\gamma_4,\gamma_3)=(\gamma_4-\gamma_3)$ in (20), we get (6), which is Theorem 2 in [8].*

Theorem 2.2. *Let $K_1 \times K_2 \subseteq \mathbb{R}^2$ be an open invex subset with respect to $\eta : K_1 \times K_1 \to \mathbb{R}^2$ and $\xi : K_2 \times K_2 \to \mathbb{R}^2$. Suppose $f : K_1 \times K_2 \to \mathbb{R}^2$ is a twice differentiable mapping on $K_1 \times K_2$ such that $f_{st} \in L([\gamma_1, \gamma_1+\eta(\gamma_2,\gamma_1)] \times ([\gamma_3,\gamma_3+\xi(\gamma_4,\gamma_3)]))$ where $\eta(\gamma_2,\gamma_1) > 0$ and $\xi(\gamma_4,\gamma_3) > 0$. If $w : ([\gamma_1,\gamma_1+\eta(\gamma_2,\gamma_1)] \times ([\gamma_3,\gamma_3+\xi(\gamma_4,\gamma_3)])) \to [0,\infty)\times[0,\infty)$ is an integral mapping and $|f_{st}|^q$ is preinvex function then for every $\gamma_1,\gamma_2 \in K_1$ and $\gamma_3,\gamma_4 \in K_2$, where $q \in (1,\infty)$ and $\frac{1}{p}+\frac{1}{q}=1$, then following inequality holds:*

$$|I| \le \frac{[\eta(\gamma_2,\gamma_1)]^{\rho+2}[\xi(\gamma_4,\gamma_3)]^{\sigma+2}}{\Gamma(\rho)\Gamma(\sigma)}\left(\frac{\|w\|_\infty^p}{2^{\rho p+\sigma p}\rho\sigma(1+\sigma p)(1+\rho p)}\right)^{\frac{1}{p}}$$
$$\times\left(\frac{1}{4}\left[\left|f_{st}(\gamma_1,\gamma_3)\right|^q + \left|f_{st}(\gamma_1,\gamma_4)\right|^q + \left|f_{st}(\gamma_2,\gamma_3)\right|^q + \left|f_{st}(\gamma_2,\gamma_4)\right|^q\right]\right)^{\frac{1}{q}}. \tag{27}$$

Proof. Taking modulus of both sides of (10) and Hölder's integral inequality, we get

$$|I| \le \frac{[\eta(\gamma_2,\gamma_1)]^{\rho+2}[\xi(\gamma_4,\gamma_3)]^{\sigma+2}}{\Gamma(\rho)\Gamma(\sigma)}\left(\int_0^1\int_0^1 |K(s,t)|^p\, dsdt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left(\int_0^1 \left|f_{st}(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))\right|^q ds\right)dt\right)^{\frac{1}{q}}. \tag{28}$$

From preinvexity of $|f_{st}|^q$ in coordinates in (28), we have

$$|I| \le \frac{[\eta(\gamma_2,\gamma_1)]^{\rho+2}[\xi(\gamma_4,\gamma_3)]^{\sigma+2}}{\Gamma(\rho)\Gamma(\sigma)}\left(\int_0^1\int_0^1 |K(s,t)|^p\, dsdt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left(\int_0^1 (1-s)(1-t)\left|f_{st}(\gamma_1,\gamma_3)\right|^q + (1-s)t\left|f_{st}(\gamma_1,\gamma_4)\right|^q\right.\right.$$
$$\left. +(1-t)s\left|f_{st}(\gamma_2,\gamma_3)\right|^q + st\left|f_{st}(\gamma_2,\gamma_4)\right|^q\right) dt)^{\frac{1}{q}}. \tag{29}$$

After a straightforward computation, we obtain

$$\begin{aligned}
&\int_0^1\int_0^1 [(1-t)(1-s)\ |f_{st}(\gamma_1,\gamma_3)|^q + (1-s)t\,|f_{st}(\gamma_1,\gamma_4)|^q \\
&\qquad +(1-t)s\,|f_{st}(\gamma_2,\gamma_3)|^q + ts\,|f_{st}(\gamma_2,\gamma_4)|^q]\,dsdt \\
&= \frac{1}{4}\left[|g_{st}(\alpha,\gamma)|^q + |g_{st}(\alpha,\delta)|^q + |g_{st}(\beta,\gamma)|^q + |g_{st}(\beta,\delta)|^q\right], \qquad (30)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1\int_0^1 |K(s,t)|^p\,dsdt &= \frac{\|w\|_\infty^p}{\rho\sigma}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^{\sigma p}s^{\rho p}dsdt + \frac{\|w\|_\infty^p}{\rho\sigma}\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 t^{\sigma p}(1-s)^{\rho p}\,dsdt \\
&+\frac{\|w\|_\infty^p}{\rho\sigma}\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} (1-t)^{\sigma p}s^{\rho p}dsdt + \frac{\|w\|_\infty^p}{\rho\sigma}\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 (1-t)^{\sigma p}(1-s)^{\rho p}\,dsdt \\
&= \frac{\|w\|_\infty^p}{2^{\rho p+\sigma p}\rho\sigma(1+\sigma p)(1+\rho p)}. \qquad (31)
\end{aligned}$$

Using the results of (30) and (31) in (29), we get required result (27). □

Corollary 2.3. *If $f=1$, $\rho=\sigma=1$, $\eta(\gamma_2,\gamma_1)=\gamma_2-\gamma_1$ and $\xi(\gamma_4,\gamma_3)=(\gamma_4-\gamma_3)$ in (27), we get (7), which is Theorem 3 in [8].*

Theorem 2.3. *Let $K_1\times K_2\subseteq\mathbb{R}^2$ be an open invex subset with respect to $\eta: K_1\times K_1\to\mathbb{R}^2$ and $\xi: K_2\times K_2\to\mathbb{R}^2$. Suppose $f: K_1\times K_2\to\mathbb{R}^2$ is a twice differentiable mapping on $K_1\times K_2$ such that $f_{st}\in L([\gamma_1,\gamma_1+\eta(\gamma_2,\gamma_1)]\times([\gamma_3,\gamma_3+\xi(\gamma_4,\gamma_3)]))$ where $\eta(\gamma_2,\gamma_1)>0$ and $\xi(\gamma_4,\gamma_3)>0$. If $w: ([\gamma_1,\gamma_1+\eta(\gamma_2,\gamma_1)]\times([\gamma_3,\gamma_3+\xi(\gamma_4,\gamma_3)]))\to[0,\infty)\times[0,\infty)$ is an integral mapping and $|f_{st}|^q$ is preinvex function then for every $\gamma_1,\gamma_2\in K_1$ and $\gamma_3,\gamma_4\in K_2$, where $q\geq 1$, then following inequality holds:*

$$\begin{aligned}
|I| &\leq \frac{[\eta(\gamma_2,\gamma_1)]^{\rho+2}[\xi(\gamma_4,\gamma_3)]^{\sigma+2}}{\Gamma(\rho)\Gamma(\sigma)}\left(\frac{\|w\|_\infty^p}{2^{\rho+\sigma}\rho\sigma(1+\sigma)(1+\rho)}\right)^{1-\frac{1}{q}} \\
&\left(\frac{1}{\rho\sigma(\rho+1)(\sigma+1)2^{2+\rho+\sigma}}\left[|f_{st}(\gamma_1,\gamma_3)|^q + |f_{st}(\gamma_1,\gamma_4)|^q + |f_{st}(\gamma_2,\gamma_3)|^q + |f_{st}(\gamma_2,\gamma_4)|^q\right]\right)^{\frac{1}{q}} (32)
\end{aligned}$$

Proof. Taking modulus of both sides of (10) and power mean inequality, we get

$$\begin{aligned}
|I| \quad &\leq \quad \frac{[\eta(\gamma_2,\gamma_1)]^{\rho+2}[\xi(\gamma_4,\gamma_3)]^{\sigma+2}}{\Gamma(\rho)\Gamma(\sigma)}\left(\int_0^1\int_0^1 |K(s,t)|\,dsdt\right)^{1-\frac{1}{q}} \\
&\left(\int_0^1\left(\int_0^1 K(s,t)\,|f_{st}(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))|^q\,ds\right)dt\right)^{\frac{1}{q}}. \qquad (33)
\end{aligned}$$

From preinvexity of $|f_{st}|^q$ in coordinates, we have from (33)

$$\begin{aligned}
J \quad &\leq \quad \frac{[\eta(\gamma_2,\gamma_1)]^{\rho+2}[\xi(\gamma_4,\gamma_3)]^{\sigma+2}}{\Gamma(\rho)\Gamma(\sigma)}\left(\int_0^1\int_0^1 |K(s,t)|\,dsdt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1\left(\int_0^1 |K(s,t)|\,(1-s)(1-t)\,|f_{st}(\gamma_1,\gamma_3)|^q + (1-s)t\,|f_{st}(\gamma_1,\gamma_4)|^q\right.\right. \\
&+(1-t)s\,|f_{st}(\gamma_2,\gamma_3)|^q + st\,|f_{st}(\gamma_2,\gamma_4)|^q)\,dt)^{\frac{1}{q}}. \qquad (34)
\end{aligned}$$

After a straightforward computation, we obtained

$$\begin{aligned}
&\|w\|_{\infty}\int_{0}^{\frac{1}{2}}\int_{0}^{\frac{1}{2}}\left(\int_{0}^{t}\int_{0}^{s}v^{\sigma-1}u^{\rho-1}dudv\left[(1-s)(1-t)\left|f_{st}(\gamma_1,\gamma_3)\right|+(1-s)tf\left|_{st}(\gamma_1,\gamma_4)\right|\right.\right.\\
&\left.\left.+(1-t)s\left|f_{st}(\gamma_2,\gamma_3)\right|+st\left|f_{st}(\gamma_2,\gamma_4)\right|\right]\right)dsdt\\
&=\frac{\|w\|_{\infty}}{2^{\rho+\sigma+4}\rho\sigma}\left[\frac{(\rho+3)(\sigma+3)}{(\rho^2+3\rho+2)(\sigma^2+3\sigma+2)}\left|f_{st}(\gamma_1,\gamma_3)\right|+\frac{(\rho+3)}{(\rho^2+3\rho+2)(\sigma+2)}\left|f_{st}(\gamma_1,\gamma_4)\right|\right.\\
&\left.+\frac{(\sigma+3)}{(\rho+2)(\sigma^2+3\sigma+2)}\left|f_{st}(\gamma_2,\gamma_3)\right|+\frac{1}{(\rho+2)(\sigma+2)}\left|f_{st}(\gamma_2,\gamma_4)\right|\right],
\end{aligned}\tag{35}$$

$$\begin{aligned}
&\int_{0}^{\frac{1}{2}}\int_{\frac{1}{2}}^{1}\left(\int_{0}^{t}\int_{s}^{1}v^{\sigma-1}(1-u)^{\rho-1}dudv\left[(1-s)(1-t)\left|f_{st}(\gamma_1,\gamma_3)\right|+(1-s)t\left|f_{st}(\gamma_1,\gamma_4)\right|\right.\right.\\
&\left.\left.+(1-t)s\left|f_{st}(\gamma_2,\gamma_3)\right|+st\left|f_{st}(\gamma_2,\gamma_4)\right|\right]\right)dsdt\\
&=\frac{\|w\|_{\infty}}{2^{\rho+\sigma+4}\rho\sigma}\left[\frac{(\sigma+3)}{(\rho+2)(\sigma^2+3\sigma+2)}\left|f_{st}(\gamma_1,\gamma_3)\right|+\frac{1}{(\rho+2)(\sigma+2)}\left|f_{st}(\gamma_1,\gamma_4)\right|\right.\\
&\left.+\frac{(\rho+3)(\sigma+3)}{(\rho^2+3\rho+2)(\sigma^2+3\sigma+2)}\left|f_{st}(\gamma_2,\gamma_3)\right|+\frac{(\rho+3)}{(\rho^2+3\rho+2)(\sigma+2)}\left|f_{st}(\gamma_2,\gamma_4)\right|\right],
\end{aligned}\tag{36}$$

$$\begin{aligned}
&\|w\|_{\infty}\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}}\left(\int_{t}^{1}\int_{0}^{s}u^{\rho-1}(1-v)^{\sigma-1}dudv\left[(1-s)(1-t)\left|f_{st}(\gamma_1,\gamma_3)\right|+(1-s)tf\left|_{st}(\gamma_1,\gamma_4)\right|\right.\right.\\
&\left.\left.+(1-t)s\left|f_{st}(\gamma_2,\gamma_3)\right|+st\left|f_{st}(\gamma_2,\gamma_4)\right|\right]\right)dsdt\\
&=\frac{\|w\|_{\infty}}{2^{\rho+\sigma+4}\rho\sigma}\left[\frac{(\rho+3)}{(\rho^2+3\rho+2)(2+\sigma)}\left|f_{st}(\gamma_1,\gamma_3)\right|+\frac{(\rho+3)(\sigma+3)}{(\rho^2+3\rho+2)(\sigma^2+3\sigma+2)}\left|f_{st}(\gamma_1,\gamma_4)\right|\right.\\
&\left.+\frac{1}{(\rho+2)(\sigma+2)}\left|f_{st}(\gamma_2,\gamma_3)\right|+\frac{(\rho+3)}{(\rho+2)(\sigma^2+3\sigma+2)}\left|f_{st}(\gamma_2,\gamma_4)\right|\right],
\end{aligned}\tag{37}$$

$$\begin{aligned}
&\|w\|_{\infty}\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\left(\int_{t}^{1}\int_{s}^{1}(1-v)^{\sigma-1}(1-u)^{\rho-1}dudv\left[(1-s)(1-t)\left|f_{st}(\gamma_1,\gamma_3)\right|+(1-s)tf\left|_{st}(\gamma_1,\gamma_4)\right|\right.\right.\\
&\left.\left.+(1-t)s\left|f_{st}(\gamma_2,\gamma_3)\right|+st\left|f_{st}(\gamma_2,\gamma_4)\right|\right]\right)dsdt\\
&=\frac{\|w\|_{\infty}}{2^{\rho+\sigma+4}\rho\sigma}\left[\frac{1}{(\rho+2)(\sigma+2)}\left|f_{st}(\gamma_1,\gamma_3)\right|+\frac{(3+\sigma)}{(\rho+2)(\sigma^2+3\sigma+2)}\left|f_{st}(\gamma_1,\gamma_4)\right|\right.\\
&\left.+\frac{(\rho+3)}{(\rho^2+3\rho+2)(\sigma+2)}\left|f_{st}(\gamma_2,\gamma_3)\right|+\frac{(\rho+3)(\sigma+3)}{(\rho^2+3\rho+2)(\sigma^2+3\sigma+2)}\left|f_{st}(\gamma_2,\gamma_4)\right|\right].
\end{aligned}\tag{38}$$

$$\int_{0}^{1}\int_{0}^{1}\left|K(s,t)\right|dsdt=\frac{\|w\|_{\infty}^{p}}{2^{\rho+\sigma}\rho\sigma(\sigma+1)(\rho+1)}.\tag{39}$$

Using the results of (35), (36), (37) , (38) and (39) in (34), we get required result. □

Corollary 2.4. *If $f = 1, \rho = \sigma = 1, \eta(\gamma_2, \gamma_1) = \gamma_2 - \gamma_1$ and $\xi(\gamma_4, \gamma_3) = (\gamma_4 - \gamma_3)$ in (32), we get (8), which is Theorem 4 in [8].*

Lemma 2.2. *Let $K_1 \times K_2 \subseteq \mathbb{R}^2$ be an open invex subset with respect to $\eta : K_1 \times K_1 \to \mathbb{R}^2$ and $\xi : K_2 \times K_2 \to \mathbb{R}^2$. Suppose $f : K_1 \times K_2 \to \mathbb{R}^2$ is a twice differentiable mapping on $K_1 \times K_2$ such that $f_{st} \in L([\gamma_1, \gamma_1 + \eta(\gamma_2, \gamma_1)] \times ([\gamma_3, \gamma_3 + \xi(\gamma_4, \gamma_3)]))$ where $\eta(\gamma_2, \gamma_1) > 0$ and $\xi(\gamma_4, \gamma_3) > 0$. If $w : ([\gamma_1, \gamma_1 + \eta(\gamma_2, \gamma_1)] \times ([\gamma_3, \gamma_3 + \xi(\gamma_4, \gamma_3)])) \to [0, \infty) \times [0, \infty)$ is an integral mapping, then for every $\gamma_1, \gamma_2 \in K_1$ and $\gamma_3, \gamma_4 \in K_2$, the following equality holds:*

$$
\begin{aligned}
J \;=\; & \Big[f(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3 + \xi(\gamma_4, \gamma_3)) I^{\rho,\sigma}_{(\gamma_1+\eta(\gamma_2,\gamma_1))-,(\gamma_3+\xi(\gamma_4,\gamma_3))-} w(\gamma_1, \gamma_3) \\
& + f(\gamma_1, \gamma_3 + \xi(\gamma_4, \gamma_3)) I^{\rho,\sigma}_{\gamma_1+,(\gamma_3+\xi(\gamma_4,\gamma_3))-} w(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3) \\
& + f(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3) I^{\rho,\sigma}_{(\gamma_1+\eta(\gamma_2,\gamma_1))-,\gamma_3+} w(\gamma_1, \gamma_3 + \xi(\gamma_4, \gamma_3)) \\
& + f(\gamma_1, \gamma_3) I^{\rho,\sigma}_{\gamma_1+,\gamma_3+} w(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3 + \xi(\gamma_4, \gamma_3))\Big] \\
& - \Big[I^{\rho,\sigma}_{(\gamma_1+\eta(\gamma_2,\gamma_1))-,(\gamma_3+\xi(\gamma_4,\gamma_3))-} f(\gamma_1, \gamma_3 + \xi(\gamma_4, \gamma_3))\, w(\gamma_1, \gamma_3) \\
& + I^{\rho,\sigma}_{\gamma_1+,(\gamma_3+\xi(\gamma_4,\gamma_3))-} f(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3 + \xi(\gamma_4, \gamma_3))\, w(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3) \\
& + I^{\rho,\sigma}_{(\gamma_1+\eta(\gamma_2,\gamma_1))-,\gamma_3+} f(\gamma_1, \gamma_3)\, w(\gamma_1, \gamma_3 + \xi(\gamma_4, \gamma_3)) \\
& + I^{\rho,\sigma}_{\gamma_1+,\gamma_3+} f(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3)\, w(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3 + \xi(\gamma_4, \gamma_3))\Big] \\
& - \Big[I^{\rho,\sigma}_{(\gamma_1+\eta(\gamma_2,\gamma_1))-,(\gamma_3+\xi(\gamma_4,\gamma_3))-} f(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3)\, w(\gamma_1, \gamma_3) \\
& + I^{\rho,\sigma}_{\gamma_1+,(\gamma_3+\xi(\gamma_4,\gamma_3))-} f(\gamma_1, \gamma_3)\, w(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3) \\
& + I^{\rho,\sigma}_{(\gamma_1+\eta(\gamma_2,\gamma_1))-,\gamma_3+} f(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3 + \xi(\gamma_4, \gamma_3))\, w(\gamma_1, \gamma_3 + \xi(\gamma_4, \gamma_3)) \\
& + I^{\rho,\sigma}_{\gamma_1+,\gamma_3+} f(\gamma_1, \gamma_3 + \xi(\gamma_4, \gamma_3))\, w(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3 + \xi(\gamma_4, \gamma_3))\Big] \\
& + \Big[I^{\rho,\sigma}_{(\gamma_1+\eta(\gamma_2,\gamma_1))-,(\gamma_3+\xi(\gamma_4,\gamma_3))-} (fw)(\gamma_1, \gamma_3) \\
& + I^{\rho,\sigma}_{\gamma_1+,(\gamma_3+\xi(\gamma_4,\gamma_3))-} (fw)(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3) \\
& + I^{\rho,\sigma}_{(\gamma_1+\eta(\gamma_2,\gamma_1))-,\gamma_3+} (fw)(\gamma_1, \gamma_3 + \xi(\gamma_4, \gamma_3)) \\
& + I^{\rho,\sigma}_{\gamma_1+,\gamma_3+} (fw)(\gamma_1 + \eta(\gamma_2, \gamma_1), \gamma_3 + \xi(\gamma_4, \gamma_3))\Big] \\
= \; & \frac{[\eta(\gamma_2, \gamma_1)]^{\rho+2} [\xi(\gamma_4, \gamma_3)]^{\sigma+2}}{\Gamma(\rho)\Gamma(\sigma)} \int_0^1 \int_0^1 P(s,t) f_{ts}((\gamma_1 + s\eta(\gamma_2, \gamma_1), \gamma_3 + t\xi(\gamma_4, \gamma_3)) ds dt,
\end{aligned} \tag{40}
$$

where

$$
P(s,t) = \int_0^t A_1(s,v)\,dv - \int_0^t A_2(s,v)\,dv - \int_t^1 A_3(s,v)\,dv + \int_t^1 A_4(s,v)\,dv, \quad (t,s) \in [0,1],
$$

and

$$
\begin{aligned}
A_1(s,v) &= \int_0^s u^{\rho-1} v^{\sigma-1} w(\gamma_1 + u\eta(\gamma_2, \gamma_1), \gamma_3 + v\xi(\gamma_4, \gamma_3)) du \\
A_2(s,v) &= \int_s^1 (1-u)^{\rho-1} v^{\sigma-1} w(\gamma_1 + u\eta(\gamma_2, \gamma_1), \gamma_3 + v\xi(\gamma_4, \gamma_3)) du
\end{aligned}
$$

$$\begin{aligned} A_3(s,v) &= \int_0^s u^{\rho-1}(1-v)^{\sigma-1}w(\gamma_1+u\eta(\gamma_2,\gamma_1),\gamma_3+v\xi(\gamma_4,\gamma_3))du \\ A_4(s,v) &= \int_s^1 (1-u)^{\rho-1}(1-v)^{\sigma-1}w(\gamma_1+u\eta(\gamma_2,\gamma_1),\gamma_3+v\xi(\gamma_4,\gamma_3))du. \end{aligned}$$

Proof. Expressing the integral on R.H.S in term of four integrals

$$\begin{aligned} & \int_0^1\int_0^1 P(s,t)f_{st}(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dsdt \\ =\ & \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} P(s,t)g_{st}(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dtds \\ & +\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}} P(s,t)g_{st}(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dtds \\ & +\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 P(s,t)g_{st}(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dtds \\ & +\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 P(s,t)g_{st}(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dtds \\ =\ & J_1+J_2+J_3+J_4. \end{aligned} \tag{41}$$

For computing J_1, we will first consider

$$\begin{aligned} & \int_0^1\left(\int_0^t A_1(s,v)\,dv\ f_{st}(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))\right)dt \\ =\ & \frac{1}{\xi(\gamma_4,\gamma_3)}\left|f_s(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))\int_0^t A_1(s,v)\,dv\right|_0^1 \\ & -\frac{1}{\xi(\gamma_4,\gamma_3)}\int_0^1 (A_1(s,t)\,f_s(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3)))\,dt \\ =\ & \frac{1}{\xi(\gamma_4,\gamma_3)}f_s(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+\xi(\gamma_4,\gamma_3))\int_0^1 A_1(s,v)\,dv \\ & -\frac{1}{\xi(\gamma_4,\gamma_3)}\int_0^1 (A_1(s,t)\,f_s(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3)))\,dt. \end{aligned} \tag{42}$$

Integrating (42) w.r.t s, we will get I_1

$$\begin{aligned} & J_1 \\ =\ & \int_0^1\int_0^1\int_0^t A_1(s,v)\,dv\ f_{st}(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dtds \\ =\ & \frac{1}{\xi(\gamma_4,\gamma_3)}\int_0^1 f_s(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+\xi(\gamma_4,\gamma_3))\int_0^1 A_1(s,v)\,dvdu \\ & -\frac{1}{\xi(\gamma_4,\gamma_3)}\int_0^1\int_0^1 (A_1(s,t)\,f_s(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3)))\,dtds. \end{aligned} \tag{43}$$

After computing the outer integral in (43), we obtain

$$J_1 = \frac{1}{\eta(\gamma_2,\gamma_1)\,\xi(\gamma_4,\gamma_3)}\left[f(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3+\xi(\gamma_4,\gamma_3))\right.$$
$$\times\int_0^1\int_0^1 s^{\rho-1}t^{\sigma-1}w(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dsdt$$
$$-\int_0^1\int_0^1 s^{\rho-1}t^{\sigma-1}f(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+\xi(_4,\gamma_3))w(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dtds$$
$$-\int_0^1\int_0^1 s^{\rho-1}t^{\sigma-1}f(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))w(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dsdt$$
$$\left.+\int_0^1\int_0^1 s^{\rho-1}t^{\sigma-1}f(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))w(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dsdt\right]. \tag{44}$$

On substituting $x=\gamma_1+s\eta(\gamma_2,\gamma_1)$ and $y=\gamma_3+t\xi(\gamma_4,\gamma_3)$ in (44),we obtain

$$\begin{aligned} J_1 \;=\; & \frac{1}{[\eta(\gamma_2,\gamma_1)]^{\rho+1}[\xi(\gamma_4,\gamma_3)]^{\sigma+1}}\left[f(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+\xi(\gamma_4,\gamma_3))\right. \\ & \times\int_{\gamma_1}^{\gamma_1+\eta(\gamma_2,\gamma_1)}\int_{\gamma_3}^{\gamma_3+\xi(\gamma_4,\gamma_3)}(x-\gamma_1)^{\rho-1}(y-\gamma_3)^{\sigma-1}w(x,y)dxdy \\ & -\int_{\gamma_1}^{\gamma_1+\eta(\gamma_2,\gamma_1)}\int_{\gamma_3}^{\gamma_3+\xi(\gamma_4,\gamma_3)}(x-\gamma_1)^{\rho-1}(y-\gamma_3)^{\sigma-1}f(x,\gamma_3+\xi(\gamma_4,\gamma_3)\;w(x,y)dxdy \\ & -\int_{\gamma_1}^{\gamma_1+\eta(\gamma_2,\gamma_1)}\int_{\gamma_3}^{\gamma_3+\xi(\gamma_4,\gamma_3)}(x-\gamma_1)^{\rho-1}(y-\gamma_3)^{\sigma-1}f(\gamma_1+\eta(\gamma_2,\gamma_1),y)\;w(x,y)dxdy \\ & \left.+\int_{\gamma_1}^{\gamma_1+\eta(\gamma_2,\gamma_1)}\int_{\gamma_3}^{\gamma_3+\xi(\gamma_4,\gamma_3)}.(x-\gamma_1)^{\rho-1}(y-\gamma_3)^{\sigma-1}f(x,y)\;w(x,y)dxdy\right]. \end{aligned} \tag{45}$$

Utilizing the definition of Riemann-Liouville fractional integrals in (45), we obtain

$$\begin{aligned} J_1 \;=\; & \frac{\Gamma(\rho)\Gamma(\sigma)}{[\eta(\gamma_2,\gamma_1)]^{\rho+1}[\xi(\gamma_4,\gamma_3)]^{\sigma+1}} \\ & \times\Big[f(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3+\xi(\gamma_4,\gamma_3))I^{\rho,\sigma}_{(\gamma_1+\eta(\gamma_2,\gamma_1))-,(\gamma_3+\xi(\gamma_4,\gamma_3))-}w(\gamma_1,\gamma_3) \\ & -I^{\rho,\sigma}_{(\gamma_1+\eta(\gamma_2,\gamma_1))-,(\gamma_3+\xi(\gamma_4,\gamma_3))-}f(\gamma_1,\gamma_3+\xi(\gamma_4,\gamma_3)\;w(\gamma_1,\gamma_3) \\ & -I^{\rho,\sigma}_{(\gamma_1+\eta(\gamma_2,\gamma_1))-,(\gamma_3+\xi(\gamma_4,\gamma_3))-}f(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3)\;w(\gamma_1,\gamma_3) \\ & +I^{\rho,\sigma}_{(\gamma_1+\eta(\gamma_2,\gamma_1))-,(\gamma_3+\xi(\gamma_4,\gamma_3))-}(fw)(\gamma_1,\gamma_3)\Big]. \end{aligned} \tag{46}$$

Similarly, we obtain

$$\begin{aligned} J_2 &= \frac{\Gamma(\rho)\Gamma(\sigma)}{[\eta(\gamma_2,\gamma_1)]^{\rho+1}[\xi(\gamma_4,\gamma_3)]^{\sigma+1}} \\ &\times\Big[-f(\gamma_1,\gamma_3+\xi(\gamma_4,\gamma_3))I^{\rho,\sigma}_{\gamma_1+,(\gamma_3+\xi(\gamma_4,\gamma_3))-}w(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3) \\ &+I^{\rho,\sigma}_{\gamma_1+,(\gamma_3+\xi(\gamma_4,\gamma_3))-}f(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3+\xi(\gamma_4,\gamma_3)\ w(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3) \\ &+I^{\rho,\sigma}_{\gamma_1+,(\gamma_3+\xi(\gamma_4,\gamma_3))-}f(\gamma_1,\gamma_3)\ w(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3) \\ &-\ I^{\rho,\sigma}_{\gamma_1+,(\gamma_3+\xi(\gamma_4,\gamma_3))-}\ (fw)(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3)\Big]. \end{aligned} \tag{47}$$

$$\begin{aligned} J_3 &= \frac{\Gamma(\rho)\Gamma(\sigma)}{[\eta(\gamma_2,\gamma_1)]^{\rho+1}[\xi(\gamma_4,\gamma_3)]^{\sigma+1}} \\ &\times\Big[-f(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3)I^{\rho,\sigma}_{(\gamma_1+\eta(\gamma_2,\gamma_1))-,\gamma_3+}w(\gamma_1,\gamma_3+\xi(\gamma_4,\gamma_3)) \\ &+I^{\rho,\sigma}_{(\gamma_1+\eta(\gamma_2,\gamma_1))-,\gamma_3+}f(\gamma_1,\gamma_3)\ w(\gamma_1,\gamma_3+\xi(\gamma_4,\gamma_3)) \\ &+I^{\rho,\sigma}_{(\gamma_1+\eta(\gamma_2,\gamma_1))-,\gamma_3+}f(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3+\xi(\gamma_4,\gamma_3))\ w(\gamma_1,\gamma_3+\xi(\gamma_4,\gamma_3)) \\ &-\ I^{\rho,\sigma}_{(\gamma_1+\eta(\gamma_2,\gamma_1))-,\gamma_3+}\ (fw)(\gamma_1,\gamma_3+\xi(\gamma_4,\gamma_3))\Big]. \end{aligned} \tag{48}$$

$$\begin{aligned} J_4 &= \frac{\Gamma(\rho)\Gamma(\sigma)}{[\eta(\gamma_2,\gamma_1)]^{\rho+1}[\xi(\gamma_4,\gamma_3)]^{\sigma+1}} \\ &\times\Big[f(\gamma_1,\gamma_3)I^{\rho,\sigma}_{\gamma_1+,\gamma_3+}w(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3+\xi(\gamma_4,\gamma_3)) \\ &-I^{\rho,\sigma}_{\gamma_1+,\gamma_3+}f(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3)\ w(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3+\xi(\gamma_4,\gamma_3)) \\ &-I^{\rho,\sigma}_{\gamma_1+,\gamma_3+}f(\gamma_1,\gamma_3+\xi(\gamma_4,\gamma_3))\ w(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3+\xi(\gamma_4,\gamma_3)) \\ &+\ I^{\rho,\sigma}_{\gamma_1+,\gamma_3+}\ (fw)(\gamma_1+\eta(\gamma_2,\gamma_1),\gamma_3+\xi(\gamma_4,\gamma_3))\big]. \end{aligned} \tag{49}$$

On utilizing (46), (47), (48) and (49) in (41), we obtain (40). □

Theorem 2.4. *Let $K_1 \times K_2 \subseteq \mathbb{R}^2$ be an open invex subset with respect to $\eta : K_1 \times K_1 \to \mathbb{R}^2$ and $\xi : K_2 \times K_2 \to \mathbb{R}^2$. Suppose $f : K_1 \times K_2 \to \mathbb{R}^2$ is a twice differentiable mapping on $K_1 \times K_2$ such that $f_{st} \in L([\gamma_1, \gamma_1 + \eta(\gamma_2, \gamma_1)] \times ([\gamma_3, \gamma_3 + \xi(\gamma_4, \gamma_3)]))$ where $\eta(\gamma_2, \gamma_1) > 0$ and $\xi(\gamma_4, \gamma_3) > 0$. If $w : ([\gamma_1, \gamma_1 + \eta(\gamma_2, \gamma_1)] \times ([\gamma_3, \gamma_3 + \xi(\gamma_4, \gamma_3)])) \to [0, \infty) \times [0, \infty)$ is an integral mapping and $|f_{st}|$ is preinvex function then for every $\gamma_1, \gamma_2 \in K_1$ and $\gamma_3, \gamma_4 \in K_2$, the following inequality holds:*

$$|J| \leq \frac{\|w\|_\infty [\eta(\gamma_2,\gamma_1)]^{\rho+2}[\xi(\gamma_4,\gamma_3)]^{\sigma+2}}{\rho\sigma(\sigma+1)(\rho+1)\Gamma(\rho)\Gamma(\sigma)}\left[|f_{st}(\gamma_1,\gamma_3)|+|f_{st}(\gamma_1,\gamma_4)|+|f_{st}(\gamma_2,\gamma_3)|+|f_{st}(\gamma_2,\gamma_4)|\right]. \tag{50}$$

Proof. Taking modulus of both sides of (40), we get

$$|J| = \left|\frac{[\eta(\gamma_2,\gamma_1)]^{\rho+2}[\xi(\gamma_4,\gamma_3)]^{\sigma+2}}{\Gamma(\rho)\Gamma(\sigma)}\int_0^1\int_0^1 P(s,t)f_{ts}(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))dsdt\right|. \tag{51}$$

f_{st} is preinvex function, i.e

$$\begin{aligned} & f_{st}(\gamma_1 + s\eta(\gamma_2,\gamma_1), \gamma_3 + t\xi(\gamma_4,\gamma_3)) \\ \leq \; & (1-s)(1-t)f_{st}(\gamma_1,\gamma_3) + (1-s)tf_{st}(\gamma_1,\gamma_4) + (1-t)sf_{st}(\gamma_2,\gamma_3) + stf_{st}(\gamma_2,\gamma_4). \end{aligned}$$

From preinvexity of $|f_{st}|$ in coordinates in (51), we have

$$\begin{aligned} |J| \leq \; & \frac{[\eta(\gamma_2,\gamma_1)]^{\rho+2}[\xi(\gamma_4,\gamma_3)]^{\sigma+2}}{\Gamma(\rho)\Gamma(\sigma)} \int_0^1 \left(\int_0^1 |P(s,t)| (1-s)(1-t)|f_{st}(\gamma_1,\gamma_3)| + (1-s)t|f_{st}(\gamma_1,\gamma_4)| \right. \\ & \left. + (1-t)s|f_{st}(\gamma_2,\gamma_3)| + st|f_{st}(\gamma_2,\gamma_4)| \right) dt. \end{aligned} \tag{52}$$

After a straightforward computations, we obtain

$$\begin{aligned} & \frac{|f_{st}(\gamma_1,\gamma_3)| \|w\|_\infty}{\rho\sigma} \int_0^1\int_0^1 s^\rho t^\sigma [ts + t(1-s) + s(1-t) + (1-t)(1-s)] \, dsdt \\ = \; & \frac{|f_{st}(\gamma_1,\gamma_3)| \|w\|_\infty}{\rho\sigma(\sigma+1)(\rho+1)}. \end{aligned} \tag{53}$$

$$\begin{aligned} & \frac{|f_{st}(\gamma_1,\gamma_4)| \|w\|_\infty}{\rho\sigma} \int_0^1\int_0^1 (1-s)^\rho t^\sigma [ts + t(1-s) + s(1-t) + (1-t)(1-s)] \, dsdt \\ = \; & \frac{|f_{st}(\gamma_1,\gamma_4)| \|w\|_\infty}{\rho\sigma(\sigma+1)(\rho+1)}. \end{aligned} \tag{54}$$

$$\begin{aligned} & \frac{|f_{st}(\gamma_2,\gamma_3)| \|w\|_\infty}{\rho\sigma} \int_0^1\int_0^1 s^\rho (1-t)^\sigma [ts + t(1-s) + s(1-t) + (1-t)(1-s)] \, dsdt \\ = \; & \frac{|f_{st}(\gamma_2,\gamma_3)| \|w\|_\infty}{\rho\sigma(\sigma+1)(\rho+1)}. \end{aligned} \tag{55}$$

$$\begin{aligned} & \frac{|f_{st}(\gamma_2,\gamma_4)| \|w\|_\infty}{\rho\sigma} \int_0^1\int_0^1 (1-s)^\rho (1-t)^\sigma [ts + t(1-s) + s(1-t) + (1-t)(1-s)] \, dsdt \\ = \; & \frac{|f_{st}(\gamma_2,\gamma_4)| \|w\|_\infty}{\rho\sigma(\sigma+1)(\rho+1)}. \end{aligned} \tag{56}$$

Using the results of (53), (54), (55) and (56) in (52), we get required result (50). □

Corollary 2.5. *If $\|f\|_\infty = 1$ and $\rho = \sigma = 1$ in (50), we get inequality similar to Theorem 2.1 in [9].*

Theorem 2.5. *Let $K_1 \times K_2 \subseteq \mathbb{R}^2$ be an open invex subset with respect to $\eta : K_1 \times K_1 \to \mathbb{R}^2$ and $\xi : K_2 \times K_2 \to \mathbb{R}^2$. Suppose $f : K_1 \times K_2 \to \mathbb{R}^2$ is a twice differentiable mapping on $K_1 \times K_2$ such that $f_{st} \in L([\gamma_1, \gamma_1 + \eta(\gamma_2,\gamma_1)] \times ([\gamma_3, \gamma_3 + \xi(\gamma_4,\gamma_3)]))$ where $\eta(\gamma_2,\gamma_1) > 0$ and $\xi(\gamma_4,\gamma_3) > 0$. If $w : ([\gamma_1, \gamma_1 + \eta(\gamma_2,\gamma_1)] \times ([\gamma_3, \gamma_3 + \xi(\gamma_4,\gamma_3)])) \to [0,\infty) \times [0,\infty)$ is an integral mapping and $|f_{st}|^q$ is preinvex function then for every $\gamma_1, \gamma_2 \in K_1$ and $\gamma_3, \gamma_4 \in K_2$, where $q \in (1,\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$, then following inequality holds:*

$$\begin{aligned} |J| \leq \; & \frac{[\eta(\gamma_2,\gamma_1)]^{\rho+2}[\xi(\gamma_4,\gamma_3)]^{\sigma+2}}{\Gamma(\rho)\Gamma(\sigma)} \left(\frac{4\|w\|_\infty^p}{\rho\sigma(1+\sigma p)(1+\rho p)} \right)^{\frac{1}{p}} \\ & \times \left(\frac{1}{4} [|g_{st}(\alpha,\gamma)|^q + |g_{st}(\alpha,\delta)|^q + |g_{st}(\beta,\gamma)|^q + |g_{st}(\beta,\delta)|^q] \right)^{\frac{1}{q}}. \end{aligned} \tag{57}$$

Proof. Taking modulus of both sides of (40) and Hölder's integral inequality, we get

$$\begin{aligned}|J| &= \frac{[\eta(\gamma_2,\gamma_1)]^{\rho+2}[\xi(\gamma_4,\gamma_3)]^{\sigma+2}}{\Gamma(\rho)\Gamma(\sigma)}\left(\int_0^1\int_0^1 |P(s,t)|^p\,dsdt\right)^{\frac{1}{p}}\\ &\quad\times\left(\int_0^1\int_0^1 |f_{ts}(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))|^q\,dsdt\right)^{\frac{1}{q}}.\end{aligned}$$

From preinvexity of $|f_{st}|^q$ in coordinates, we have

$$\begin{aligned}|J| &\le \frac{[\eta(\gamma_2,\gamma_1)]^{\rho+2}[\xi(\gamma_4,\gamma_3)]^{\sigma+2}}{\Gamma(\rho)\Gamma(\sigma)}\left(\int_0^1\int_0^1 |P(s,t)|^p\,dsdt\right)^{\frac{1}{p}}\\ &\quad\times\left(\int_0^1\left(\int_0^1(1-s)(1-t)\,|f_{st}(\gamma_1,\gamma_3)|^q+(1-s)t\,|f_{st}(\gamma_1,\gamma_4)|^q\right.\right.\\ &\quad\left.\left.+(1-t)s\,|f_{st}(\gamma_2,\gamma_3)|^q+st\,|f_{st}(\gamma_2,\gamma_4)|^q\right)dt\right)^{\frac{1}{q}}. \end{aligned} \tag{58}$$

After a straightforward computation, we obtain

$$\begin{aligned}&\int_0^1\int_0^1 [(1-s)(1-t)\,|g_{st}(\alpha,\gamma)|^q+(1-s)tg\,|_{st}(\alpha,\delta)|^q\\ &\qquad+(1-t)s\,|g_{st}(\beta,\gamma)|^q+st\,|g_{st}(\beta,\delta)|^q)\,dsdt\\ &=\frac{1}{4}\left[|g_{st}(\alpha,\gamma)|^q+|g_{st}(\alpha,\delta)|^q+|g_{st}(\beta,\gamma)|^q+|g_{st}(\beta,\delta)|^q\right],\end{aligned} \tag{59}$$

and

$$\begin{aligned}\int_0^1\int_0^1 |P(s,t)|^p\,dsdt &= \|w\|_\infty^p\int_0^1\int_0^1\frac{s^{\rho p}t^{\sigma p}}{\rho\sigma}dsdt+\|w\|_\infty^p\int_0^1\int_0^1\frac{t^{\sigma p}(1-s)^{\rho p}}{\rho\sigma}dsdt\\ &+\|w\|_\infty^p\int_0^1\int_0^1\frac{s^{\rho p}(1-t)^{\sigma p}}{\rho\sigma}dsdt+\|w\|_\infty^p\int_0^1\int_0^1\frac{(1-s)^{\rho p}(1-t)^{\sigma p}}{\rho\sigma}dsdt\\ &=\frac{4\|w\|_\infty^p}{\rho\sigma(1+\sigma p)(1+\rho p)}.\end{aligned} \tag{60}$$

Using the results of (59) and (60) in (58), we get required result (57). □

Corollary 2.6. *If $\|f\|_\infty = 1$ and $\rho=\sigma=1$ in (57), we get inequality similar to Theorem 2.2 in [9].*

Theorem 2.6. *Let $K_1\times K_2\subseteq\mathbb{R}^2$ be an open invex subset with respect to $\eta : K_1\times K_1\to\mathbb{R}^2$ and $\xi : K_2\times K_2\to\mathbb{R}^2$. Suppose $f : K_1\times K_2\to\mathbb{R}^2$ is a twice differentiable mapping on $K_1\times K_2$ such that $f_{st}\in L([\gamma_1,\gamma_1+\eta(\gamma_2,\gamma_1)]\times([\gamma_3,\gamma_3+\xi(\gamma_4,\gamma_3)]))$ where $\eta(\gamma_2,\gamma_1)>0$ and $\xi(\gamma_4,\gamma_3)>0$. If $w : ([\gamma_1,\gamma_1+\eta(\gamma_2,\gamma_1)]\times([\gamma_3,\gamma_3+\xi(\gamma_4,\gamma_3)]))\to[0,\infty)\times[0,\infty)$ is an integral mapping and $|f_{st}|^q$ is preinvex function then for every $\gamma_1,\gamma_2\in K_1$ and $\gamma_3,\gamma_4\in K_2$, where $q\ge 1$, then following inequality holds:*

$$\begin{aligned}|J| &\le \frac{[\eta(\gamma_2,\gamma_1)]^{\rho+2}[\xi(\gamma_4,\gamma_3)]^{\sigma+2}}{\Gamma(\rho)\Gamma(\sigma)}\left(\frac{4\|w\|_\infty}{\rho\sigma(\sigma+1)(\rho+1)}\right)^{1-\frac{1}{q}}\\ &\quad\times\left(\frac{\|w\|_\infty}{(1+\sigma)(\rho\sigma+\rho^2\sigma)}\left[|f_{st}(\gamma_1,\gamma_3)|^q+|f_{st}(\gamma_1,\gamma_4)|^q+|f_{st}(\gamma_2,\gamma_3)|^q+|f_{st}(\gamma_2,\gamma_4)|^q\right]\right)^{\frac{1}{q}}.\end{aligned} \tag{61}$$

Proof. Taking modulus of both sides of (40) and power mean inequality, we get

$$\begin{aligned}|J| &= \frac{[\eta(\gamma_2,\gamma_1)]^{\rho+2}[\xi(\gamma_4,\gamma_3)]^{\sigma+2}}{\Gamma(\rho)\Gamma(\sigma)}\left(\int_0^1\int_0^1|P(s,t)|\,dsdt\right)^{1-\frac{1}{q}}\\ &\quad\times\left(\int_0^1\int_0^1|P(s,t)|\,|f_{ts}(\gamma_1+s\eta(\gamma_2,\gamma_1),\gamma_3+t\xi(\gamma_4,\gamma_3))|^q\,dsdt\right)^{\frac{1}{q}}.\end{aligned}$$

From preinvexity of $|f_{st}|^q$ in coordinates, we have

$$\begin{aligned}|J| &\leq \frac{[\eta(\gamma_2,\gamma_1)]^{\rho+2}[\xi(\gamma_4,\gamma_3)]^{\sigma+2}}{\Gamma(\rho)\Gamma(\sigma)}\left(\int_0^1\int_0^1|P(s,t)|\,dsdt\right)^{1-\frac{1}{q}}\\ &\quad\times\left(\int_0^1\left(\int_0^1|P(s,t)|\,(1-s)(1-t)\,|f_{st}(\gamma_1,\gamma_3)|^q+(1-s)t\,|f_{st}(\gamma_1,\gamma_4)|^q\right.\right.\\ &\quad\left.\left.+(1-t)s\,|f_{st}(\gamma_2,\gamma_3)|^q+st\,|f_{st}(\gamma_2,\gamma_4)|^q\right)dt\right)^{\frac{1}{q}}.\end{aligned} \tag{62}$$

After a straightforward computation, we obtain

$$\begin{aligned}&\frac{|f_{st}(\gamma_1,\gamma_3)|^q\,\|w\|_\infty}{\rho\sigma}\int_0^1\int_0^1 s^\rho t^\sigma\,[ts+t(1-s)+s(1-t)+(1-t)(1-s)]\,dsdt\\ &=\frac{|f_{st}(\gamma_1,\gamma_3)|^q\,\|w\|_\infty}{\rho\sigma(\sigma+1)(\rho+1)}.\end{aligned} \tag{63}$$

$$\begin{aligned}&\frac{|f_{st}(\gamma_1,\gamma_4)|^q\,\|w\|_\infty}{\rho\sigma}\int_0^1\int_0^1 (1-s)^\rho t^\sigma\,[ts+t(1-s)+s(1-t)+(1-t)(1-s)]\,dsdt\\ &=\frac{|f_{st}(\gamma_1,\gamma_4)|^q\,\|w\|_\infty}{\rho\sigma(\sigma+1)(\rho+1)}.\end{aligned} \tag{64}$$

$$\begin{aligned}&\frac{|f_{st}(\gamma_2,\gamma_3)|^q\,\|w\|_\infty}{\rho\sigma}\int_0^1\int_0^1 s^\rho (1-t)^\sigma\,[ts+t(1-s)+s(1-t)+(1-t)(1-s)]\,dsdt\\ &=\frac{|f_{st}(\gamma_2,\gamma_3)|^q\,\|w\|_\infty}{\rho\sigma(\sigma+1)(\rho+1)}.\end{aligned} \tag{65}$$

$$\begin{aligned}&\frac{|f_{st}(\gamma_2,\gamma_4)|^q\,\|w\|_\infty}{\rho\sigma}\int_0^1\int_0^1 (1-s)^\rho (1-t)^\sigma\,[ts+t(1-s)+s(1-t)+(1-t)(1-s)]\,dsdt\\ &=\frac{|f_{st}(\gamma_2,\gamma_4)|^q\,\|w\|_\infty}{\rho\sigma(\sigma+1)(\rho+1)}.\end{aligned} \tag{66}$$

and

$$\int_0^1\int_0^1|P(s,t)|\,dsdt=\frac{4\,\|w\|_\infty}{\rho\sigma(\sigma+1)(\rho+1)}. \tag{67}$$

Using the results of (63), (64), (65), (66) and (67) in (62), we get required result. □

Corollary 2.7. *If* $\|f\|_\infty = 1$ *and* $\rho=\sigma=1$ *in* (61), *we get inequality similar to Theorem 2.3 in* [9].

3 Conclusions

Some new estimates for the lower and upper boundaries of fractional Hermite-Hadamard-Fejér type inequality are obtained for coordinated preinvex functions which add up to the literature new error bounds for the lower and upper boundaries of the Hermite-Hadamard-Fejér inequality for preinvex functions in fractional integrals correspondingly.

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Conflicts of Interest
The authors declare no conflict of interest.

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